Def. 1: Let $G \subset \mathbb{R}^m$ be a subset, and let $p \in G$. The *tangent space* to G at p is:

$T_p G = \{ \gamma'(0) \mid \gamma : (-\varepsilon, \varepsilon) \to G$ is differentiable with $\gamma(0) = p \}$.

Def. 2: The *Lie algebra* of a matrix group $G \subset GL_n(\mathbb{K})$ is the tangent space to G at Id.

Notation: $\mathfrak{g} = \mathfrak{g}(G) = T_{Id} G$.

Prop. 1 (product rule): If γ, β are differentiable paths, then so is the product path

$(\gamma \cdot \beta)(t) = \gamma(t) \cdot \beta(t)$, and

$(\gamma \cdot \beta)'(t) = \gamma(t) \cdot \beta'(t) + \gamma'(t) \cdot \beta(t)$

Prop. 2: The Lie algebra $\mathfrak{g}$ of a matrix group $G \subset GL_n(\mathbb{K})$ is a real subspace of $M_n(\mathbb{K})$.

Def. 3: The *dimension* of a matrix group G means the dimension of its Lie algebra.

Lie algebras of the orthogonal groups

The set

$o_n(K) = \{A \in M_n(K) \mid A + A^* = 0\}$ is denoted $\dim_{\mathbb{R}} o_n(k) = \binom{n}{2}$

- $so(n)$ and called the <u>skew-symmetric</u> matrices if $K = \mathbb{R}$ $\qquad \dim_{\mathbb{R}} SO(k) = \binom{n}{2}$
- $u(n)$ and called the <u>skew-hermitian</u> matrices if $K = \mathbb{C}$ $\qquad \dim_{\mathbb{C}} u(n) = n^2$
- $su(n)$ and called the skew-hermitian matrices <u>with trace 0</u> $\qquad \dim_{\mathbb{C}} SU(n) = n^2 - 1$
- $gl(n)$ and equals $M_n(K)$ $\qquad \dim gl(n) = n^2$
- $sl(n)$ and equals $M_n(K)$ with trace 0. $\qquad \dim sl(n) = n^2 - 1$

Proof of Lie Algebras

$Gl_n(\mathbb{R}), Sl_n(\mathbb{R})$: see p. 76 of Baker

$O(n), SO(n)$: see p. 81, 82 of Baker

$u(n), SU(n)$: see p. 83 of Baker

Lie Algebra vectors as vector fields

A vector field on $\mathbb{R}^m$ means a continuous function $F: \mathbb{R}^m \longrightarrow \mathbb{R}^m$.

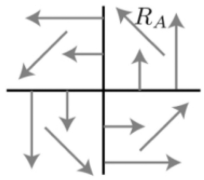
By picturing $F(v)$ as a vector drawn at $v \in \mathbb{R}^m$, we thing of a vector field as associating a vector to each point of $\mathbb{R}^m$.

Example: Lie Algebra vector of $SO(2)$

path:

$$\gamma(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$$A = \gamma'(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



$A \in so(2)$

Complexification

Def. Given a finite dimensional $\mathbb{R}$-Lie Algebra g, a $\mathbb{C}$-Lie algebra g' which contains g as an $\mathbb{R}$-Lie subalgebra and for which $\dim_{\mathbb{C}} g' = \dim_{\mathbb{R}} g$ is called a complexification of g.

Exercise 1: Prove that the Lie Algebra $u(1)$ of $U(1)$ equals $\text{span}_{\mathbb{C}}\{(i)\}$, so $\dim_{\mathbb{C}}(U(1)) = 1$.

Exercise 2: Show that $su(2)_{\mathbb{C}} \cong sl_2(\mathbb{C})$.